Dynamics of absorption of a randomly accelerated particle

2000 J. Phys. A: Math. Gen. 33 L429

(http://iopscience.iop.org/0305-4470/33/45/102)

J. Phys. A: Math. Gen. **33** (2000) L429–L432. Printed in the UK PII: S0305-4470(00)15841-X

LETTER TO THE EDITOR

Dynamics of absorption of a randomly accelerated particle

Theodore W Burkhardt‡
Institut Laue-Langevin, BP 156, F-38042 Grenoble Cedex 9, France

Received 13 July, in final form 28 September 2000

Abstract. Consider a randomly accelerated particle moving on the half-line $x > 0$ with a boundary condition at $x = 0$ that respects the scale invariance of the equations of motion under $x \to \lambda^3 x$, $v \to \lambda v$, $t \to \lambda^2 t$. If the boundary condition leads to absorption of the particle at $x = 0$ and if the probability $Q(x, v; t)$ that the particle has not yet been absorbed at time t decays, for long times, as a power law with exponent ϕ, then the power law must have the specific form $Q(x, v; t) \approx C x^{2\phi/3} U(-\frac{2}{3}\phi, \frac{2}{3}, \frac{v^3}{9x}) t^{-\phi}$. This is a consequence of scale invariance and the Fokker–Planck equation. Here C is a constant, and $U(a, b, z)$ is a confluent hypergeometric function. The persistence exponents ϕ for several boundary conditions of physical interest follow directly from this result.

Consider a particle moving on the half-line $x > 0$ according to the Langevin equation

$$\frac{d^2x}{dt^2} = \eta(t). \tag{1}$$

The acceleration $\eta(t)$ has the form of Gaussian white noise, with

$$\langle \eta(t) \rangle = 0 \qquad \langle \eta(t_1)\eta(t_2) \rangle = 2\delta(t_1 - t_2). \tag{2}$$

Several boundary conditions leading to absorption of the particle at $x = 0$ are described below. We will be primarily interested in the survival probability $Q(x, v; t)$, i.e. the probability that after a time t a particle with initial position x and initial velocity v has still not been absorbed at the boundary. The evolution of $Q(x, v; t)$ is determined by the Fokker–Planck equation [1]

$$\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x} - \frac{\partial^2}{\partial v^2} \right) Q(x, v; t) = 0 \tag{3}$$

which follows from (1) and (2), with the initial condition

$$Q(x, v; 0) = 1. \tag{4}$$

Suppose the particle is absorbed as soon as it reaches $x = 0$. This corresponds to the boundary condition

$$Q(0, v; t) = 0 \qquad v < 0 \qquad \text{absorption at first passage.} \tag{5}$$

The long-time behaviour, studied in [2–5], is given by

$$Q(x, v; t) \approx 3^{4/3}(2\pi)^{-1/2}\Gamma(3/4)^{-1} x^{1/6} U\left(-\frac{1}{6}, \frac{2}{3}, \frac{v^3}{9x}\right) t^{-1/4}. \tag{6}$$

‡ Permanent address: Department of Physics, Temple University, Philadelphia, PA 19122, USA.

Here $U(a, b, z)$ is a confluent hypergeometric function in the notation of [6]. Evaluating $Q(x, v, t)$ in the limit $x \to 0$ using the asymptotic forms [5,6]

$$U\left(-\frac{1}{6}, \frac{2}{3}, y^3\right) \approx \begin{cases} y^{1/2}\left[1 + \mathrm{O}(y^{-3})\right] & y \to \infty \\ \text{constant} \times (-y)^{-5/2} \exp(y^3) & y \to -\infty \end{cases} \tag{7}$$

we see that the boundary condition (5) is indeed satisfied and that

$$Q(0, v; t) \approx 3(2\pi)^{-1/2}\Gamma(3/4)^{-1}\left(\frac{v^2}{t}\right)^{1/4} \qquad v > 0. \tag{8}$$

Following Majumdar and Bray [7] and Cornell *et al* [8], we also consider 'partial-survival' and 'inelastic' boundary conditions, respectively. For these boundary conditions the survival probability also decays as a power law of the form

$$Q(x, v; t) \approx A(x, v)t^{-\phi} \tag{9}$$

for long times.

One can imagine physical systems in which the absorption of the particle, once it arrives at $x = 0$, is statistical, due, for example, to a quantum capture processor or to randomness of the absorption sites. In the partial-survival model [7], the randomly accelerated particle is reflected with probability p and absorbed with probability $1 - p$ each time it reaches the origin. Thus $Q(x, v; t)$ satisfies the boundary condition

$$Q(0, -v; t) = pQ(0, v; t) \qquad v > 0 \qquad \text{partial-survival.} \tag{10}$$

Burkhardt [10] and De Smedt *et al* [11] recently obtained the exact result

$$\phi(p) = \frac{1}{4}\left(1 - \frac{6}{\pi}\sin^{-1}\frac{p}{2}\right) \tag{11}$$

for the persistence exponent. Equation (11) is consistent with the expected results $\phi(0) = \frac{1}{4}$ for absorption at first passage and $\phi(1) = 0$ for elastic reflection with probability 1.

In a gas of driven granular matter the particles lose kinetic energy in inelastic collisions and tend to cluster. Under certain circumstances particles may undergo 'inelastic collapse', sticking together due to repeated inelastic collisions, even though no attractive force is present [9]. Cornell *et al* [8] have studied the inelastic collapse of a randomly accelerated particle on the half-line making inelastic collisions with the boundary. In their model the velocities v_{i} and v_{f} just before and after boundary collisions are related by $v_{\mathrm{f}} = -rv_{\mathrm{i}}$, with coefficient of restitution $r < 1$. Thus the survival probability satisfies

$$Q(0, -v; t) = Q(0, rv; t) \qquad v > 0 \qquad \text{inelastic.} \tag{12}$$

Cornell *et al* argue that there is a transition at the critical value

$$r_{\mathrm{c}} = e^{-\pi/\sqrt{3}} = 0.163\ldots. \tag{13}$$

For $r < r_{\mathrm{c}}$ the collisions are so inelastic that the particle is eventually localized, i.e. absorbed at the boundary. Swift and Bray [12] conjectured that the persistence exponent $\theta(r)$ for the inelastic model is the same as in the partial-survival model with $p = r^{2\theta}$. The exact result

$$r = \left\{2\sin\left[\frac{\pi}{6}(1 - 4\theta)\right]\right\}^{1/2\theta} \tag{14}$$

derived in [10,11] confirms this. This formula is consistent with the expected results $\theta(0) = \frac{1}{4}$ for absorption at first passage and $\theta(r_{\mathrm{c}}) = 0$ at the threshold of absorption.

Since publication of [5] I have been asked many times if the result $\phi = \frac{1}{4}$ for absorption at first passage has a simple explanation. A much simpler derivation than in the rather technical papers [3–5] is provided here. For other scale-invariant boundary conditions, including the partial-survival and inelastic cases, the derivation of the persistence exponent is equally simple.

The Fokker–Planck equation (3), the initial condition (4), and the three boundary conditions (5), (10) and (12) are all invariant under the scale transformation $x \to \lambda^3 x$, $v \to \lambda v$, $t \to \lambda^2 t$ for arbitrary positive λ. For any scale-invariant boundary condition, the survival probability

$$Q(x, v; t) = Q(\lambda^3 x, \lambda v, \lambda^2 t) = \mathcal{Q}(v^3 x^{-1}, x^{2/3} t^{-1}) \tag{15}$$

is a function of only two independent variables.

For consistency with the $t^{-\phi}$ decay, as in (9), $Q(x, v; t)$ in (15) must have the asymptotic form

$$Q(x, v; t) \approx x^{2\phi/3} G(v^3 x^{-1}) t^{-\phi} \tag{16}$$

for large t. Inserting (16) in the Fokker–Planck equation (3), one finds that the function $G(z)$ in (16) satisfies Kummer's differential equation [6]

$$zG''(z) + \left(\frac{2}{3} - \frac{z}{9}\right) G'(z) + \frac{2\phi}{27} G(z) = 0. \tag{17}$$

The general solution is a linear combination of the confluent hypergeometric functions $M(-\frac{2}{3}\phi, \frac{2}{3}, z)$ and $U(-\frac{2}{3}\phi, \frac{2}{3}, z)$, but only the latter is finite in the limit $z \to \infty$. Thus,

$$Q(x, v; t) \approx C x^{2\phi/3} U\left(-\frac{2}{3}\phi, \frac{2}{3}, \frac{v^3}{9x}\right) t^{-\phi}. \tag{18}$$

An analogous result has been obtained previously in the context of semiflexible polymers [13, 14], where the same Fokker–Planck equation plays a central role. The amplitude C in equation (18) depends on the particular boundary condition but not on x and v. For the partial-survival model the exact $C(p)$ has recently been been calculated [11, 15]. Equation (18) generalizes the result (6) for absorption at first passage to any scale-invariant boundary condition.

Taking the limit $x \to 0$ in equation (18) and making use of asymptotic properties of the confluent hypergeometric function [6], we obtain

$$Q(0, -v; t) \approx 2\sin\left[\frac{\pi}{6}(1 - 4\phi)\right] \tilde{C} \left(\frac{v^2}{t}\right)^{\phi} \qquad v > 0 \tag{19}$$

$$Q(0, v; t) \approx \tilde{C} \left(\frac{v^2}{t}\right)^{\phi} \qquad v > 0 \tag{20}$$

where $\tilde{C} = 3^{-4\phi/3} C$. If the survival probability decays as $t^{-\phi}$ and the boundary condition is scale invariant, then the value of ϕ follows directly from equations (19) and (20).

Comparing (19) and (20) with the boundary condition (5) for absorption at first passage, one sees that $\sin\left[\frac{\pi}{6}(1 - 4\phi)\right] = 0$, which implies the known exact result $\phi = \frac{1}{4}$ in (6).

Comparing (19) and (20) with the boundary condition (10) for the partial-survival model, one sees that $2\sin\left[\frac{\pi}{6}(1 - 4\phi)\right] = p$. Solving for ϕ, we obtain the known exact result (11).

Comparing (19) and (20) with the boundary condition (12) for the inelastic model and denoting the persistence exponent by θ instead of ϕ, we see that $2\sin\left[\frac{\pi}{6}(1 - 4\theta)\right] = r^{2\theta}$, which is the same as the exact result (14).

If the boundary condition is scale invariant but inconsistent with the very restrictive form (19), (20), a solution of the Fokker–Planck equation that decays as $t^{-\phi}$ can be ruled out. Are there any other scale-invariant boundary conditions besides the three we have already considered which are compatible with (19), (20)? One additional example is the generalization mentioned in [10]. The randomly accelerated particle is reflected with coefficient of restitution r_i with probability p_i, where $i = 1, 2, \ldots, n$. This corresponds to

$$Q(0, -v; t) = \sum_{i=1}^{n} p_i Q(0, r_i v; t) \qquad v > 0. \tag{21}$$

Comparing equations (19), (20) with the boundary condition (21), one sees that the persistence exponent is the same as in the partial survival model with

$$2 \sin\left[\frac{\pi}{6}(1 - 4\phi)\right] = p = \sum_{i=1}^{n} p_i r_i^{2\phi}. \tag{22}$$

An example of a boundary condition which is not invariant under $v \to \lambda v$ is

$$Q(0, -v; t) = \theta(v^* - v) Q(0, v; t) \qquad v > 0. \tag{23}$$

This corresponds to elastic reflection if the particle strikes the boundary with speed less than v^* and absorption if the incident speed exceeds v^*. Clearly this boundary condition is incompatible with equations (19) and (20). If the boundary condition is not scale invariant, $Q(x, v; t)$ depends, in general, on three independent variables x, v, t instead of two scale-invariant combinations, as in (15), and our main results (19)–(21) no longer apply.

I thank Lucian Anton, Dominique Bicout, Alan Bray, Bernard Derrida, Gerd Eilenberger, Claude Godrèche, Gerhard Gompper and Sidney Redner for many stimulating discussions and the Institut für Festkörperforschung, Jülich, Ecole Normale Supérieure, Paris, and Institut Laue-Langevin, Grenoble for hospitality. This work was also supported by the Alexander von Humboldt Stiftung and Temple University.

References

[1] Risken H 1989 *The Fokker–Planck Equation: Method of Solution and Applications* (Berlin: Springer)
[2] McKean H P 1963 *J. Math. Kyoto Univ.* **2** 227
[3] Goldman M 1971 *Ann. Math. Stat.* **42** 2150
[4] Sinai Y G 1992 *Theor. Math. Phys.* **90** 219 (1992 *Teor. Mat. Fiz.* **90** 323)
[5] Burkhardt T W 1993 *J. Phys. A: Math. Gen.* **26** L1157
[6] Abramowitz M and Stegun I A 1965 *Handbook of Mathematical Functions* (New York: Dover)
[7] Majumdar S N and Bray A J 1998 *Phys. Rev. Lett.* **81**, 2626
[8] Cornell S J, Swift M R and Bray A J 1998 *Phys. Rev. Lett.* **81** 1142
[9] Jaeger H M, Nagel S R and Behringer R P 1996 *Rev. Mod. Phys.* **68** 1259
[10] Burkhardt T W 2000 *Phys. Rev.* E at press
[11] De Smedt G, Godrèche C and Luck J M 2000 *Preprint* cond-mat/0009001
[12] Swift M R and Bray A J 1999 *Phys. Rev. E* **59** R4721
[13] Maggs A C, Huse D A and Leibler S 1989 *Europhys. Lett.* **8** 615
[14] Gompper G and Burkhardt T W 1989 *Phys. Rev. A* **40** 6124
[15] Burkhardt T W 2000 to be published